Graphs with girth 9 and without longer odd holes are 3-colorable

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Coloring of Graphs

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- A k-coloring is proper if adjacent vertices have distinct colors.
- A graph is k-colorable if it has a proper k-coloring.
- $\chi(G) =: \min\{k\}$ such that G is k-colorable.

Coloring of Graphs

A *clique* in a graph G is a subgraph induced by a set of pairwise adjacent vertices.

- The size of a largest clique in G is called the *clique number* of G, and is denoted by $\omega(G)$.
- $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.
- In 1941, Brooks proved that if G is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.

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- In 1941, Brooks proved that if G is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.
- The computation of both graph parameters $\omega(G)$ and $\chi(G)$ is NP-hard.

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- Now, the interest thing is to search the hereditary family of graphs attaining equality for the clique number ω and the chromatic number χ of its members.

Perfect graphs

A graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

Berge contributed for the fascinating class of perfect graphs more than 70 years ago two inspiring conjectures: the perfect graph conjecture proven by Lovász and the strong perfect graph conjecture proven by Chudnovsky, Robertson, Seymour and Thomas.

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Strong Perfect Graph Theorem [Chudnovsky et al.,2006,Ann.]

A graph is perfect if and only if it contains neither an odd hole nor its complement.

χ-binding functions

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- Let $\mathcal{F}$ be a family of graphs. We say that G is $\mathcal{F}$-free if it does not contain any induced subgraph which is isomorphic to a graph in $\mathcal{F}$.

χ-binding functions

Two directions

- forbid acyclic subgraphs;
- forbid unlimited cycles.

Known results

A hole in a graph is an induced cycle of length at least 4. A hole is said to be *odd* (resp. *even*) if it has odd (resp. even) length.

even hole-free graphs [Addario-Berry et al., 2008, JCTB]

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Conjecture [Hoáng, 2018, JGT]

For an odd hole free graph G, $\chi(G) \leq \frac{(\omega+1)\omega}{2}$.

Known results

The girth of a graph G, denoted by $g(G)$, is the minimum length of a cycle in G.

Let $l \geq 2$ be an integer. Let $\mathcal{G}_l$ denote the family of graphs that have girth $2l+1$ and have no odd holes of length at least $2l+3$.

The graphs in $\mathcal{G}_2$ are called *pentagraphs*, and the graphs in $\mathcal{G}_3$ are called *heptagraphs*.

Known results

Conj [Plummer and Zha, 2014, EJC]

Every pentagraph is 3-colorable.

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pentagraphs [Xu et al., 2017, EJC]

$\chi(G) = 4$.

Known results

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Graphs in $\bigcup_{l\geq 2} \mathcal{G}_l$ are 4-colorable.

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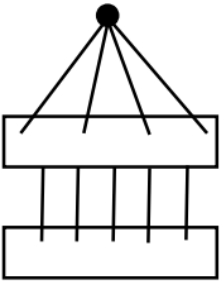

the levelling of G

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$\mathcal{G}_l, l \geq 5$ [Chen, 2024+]

All graphs in $\bigcup_{l \geq 5} \mathcal{G}_l$ are 3-colorable.

Our results

Theorem 1 [Wang and Wu, 2024+]

Graphs in $\mathcal{G}_4$ are 3-colorable.

Sketch of the proof

A parity star-cutset is a cutset of $X \subseteq V(G)$ such that X has a vertex, say x, which is adjacent to every other vertex in X, and $G - X$ has a component, say A, such that every two vertices in $X \backslash \{x\}$ are joint by an induced even path with interior in $V(A)$.

If A can be chosen such that in addition, x has a neighbour in $V(A)$, X is called a strong parity star-cutset.

Sketch of the proof

pentagraphs [Chudnovsky and Seymour, 2023, JGT]

Let G be a pentagraph. Then either

- G is a bipartite; or
- G is isomorphic to the Petersen graph; or
- G has a vertex of degree at most two; or
- G admits a P_3-cutset or a strong parity star-cutset.

heptagraphs [Wu et al., 2024+]

Let G be a heptagraph. Then either

- G is a bipartite; or
- G has a vertex of degree at most two; or
- G admits a P_3-cutset or a parity star-cutset.

Sketch of the proof

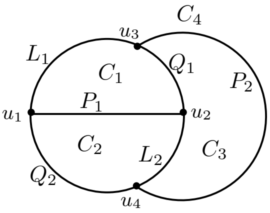
$\mathcal{G}_l, l \geq 4$ [Chen; Wang and Wu, 2024+]

Let $G \in \bigcup_{l\geq 4} \mathcal{G}_l$. Assume G has no 2-edge-cut or K_2-cut. Then one of the following holds.

1) G has an odd K_4-subdivision.
2) G contains a balanced K_4-subdivision of type $(1, 2)$.
3) G has a P_3-cut.
4) G has a vertex of degree at most two.

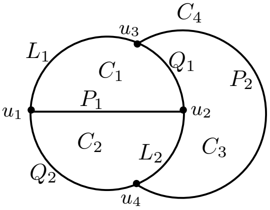
Sketch of the proof

Let $H = (u_1, u_2, u_3, u_4, P_1, P_2, Q_1, Q_2, L_1, L_2)$ be a K_4-subdivision such that u_1, u_2, u_3, u_4 are degree-3 vertices of H and P_1 is a (u_1, u_2)-path, P_2 is a (u_3, u_4)-path, Q_1 is a (u_2, u_3)-path, Q_2 is a (u_1, u_4)-path, L_1 is a (u_1, u_3)-path, and L_2 is a (u_2, u_4)-path. We call P_1, P_2, Q_1, Q_2, L_1, L_2 *arrises* of H. Let $C_1 := P_1 \cup Q_1 \cup L_1$, $C_2 := P_1 \cup Q_2 \cup L_2$, $C_3 = P_2 \cup Q_1 \cup L_2$ and $C_4 := C_1 \triangle C_2 \triangle C_3$ be four holes in H.



H

Sketch of the proof



H

We call that H is an *odd K_4-subdivision* if C_1, C_2, C_3 and C_4 are odd holes. If C_1 and C_2 are odd holes, C_3 and C_4 are even holes, $|Q_1| = 1$ and $|L_2| \geq 2$, then we call H a *balanced K_4-subdivision of type* $(1, 2)$.

Sketch of the proof

Lemma 1 [Chen, 2024+]

For any number $k \geq 4$, each k-vertex-critical graph has no 2-edge-cut.

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Lemma 2 [Chudnovsky and Seymour, 2023, JGT]

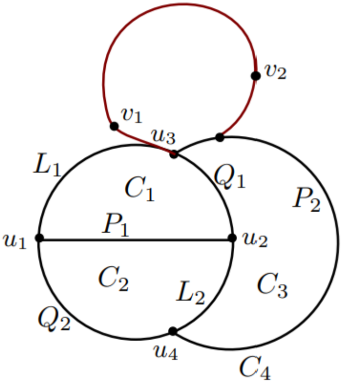
For any number $l \geq 2$, every 4-vertex-critical graph in $\mathcal{G}_l$ has neither K_2-cut or P_3-cut.

Sketch of the proof

Lemma 3 [Wang and Wu, 2024+]

Let $l \geq 4$ be an integer. For each graph G in $\mathcal{G}_l$, suppose G is 4-vertex-critical, either G has no odd K_4-subdivision or G has an odd K_4-subdivision $H = (u_1, u_2, u_3, u_4, P_1, P_2, Q_1, Q_2, L_1, L_2)$ such that every minimal direct connection (v_1, v_2)-path linking $H \backslash P_2^*$ and P_2^* must have $N_H(v_1) = N_{H \backslash P_2^*}(v_1) = \{u_3\}$ or $\{u_4\}$ and $N_H(v_2) = N_{P_2^*}(v_2) = N_{P_2^*}(N_H(v_1))$.

Sketch of the proof



H with its direct connection

Sketch of the proof

Theorem 1 [Wang and Wu, 2024+]

Let $l \geq 4$ be an integer. For each graph G in $\mathcal{G}_l$, if G is 4-vertex-critical, then G has no odd K_4-subdivisions.

Sketch of the proof

Theorem 1 [Wang and Wu, 2024+]

Let $l \geq 4$ be an integer. For each graph G in $\mathcal{G}_l$, if G is 4-vertex-critical, then G has no odd K_4-subdivisions.

Lemma 4 [Chen, 2024+]

Let $l \geq 4$ be an integer and G be a graph in $\mathcal{G}_l$. If G is 4-vertex-critical, then G does not contain a balanced K_4-subdivision of type $(1, 2)$.

Sketch of the proof

Theorem 2 [Wang and Wu, 2024+]

Let $G \in \mathcal{G}_4$. Assume G has no 2-edge-cut or K_2-cut. Then one of the following holds.

1) G has an odd K_4-subdivision.
2) G contains a balanced K_4-subdivision of type $(1, 2)$.
3) G has a P_3-cut.
4) G has a vertex of degree at most two.

It is clearly that G contains a hole C. The key to find the P_3-cut of G is relabeling the indicies of $V(C)$.

Open problems

Conj [Chen, 2024+]

For an integer $l \geq 2$ and a graph G with $g(G) = 2l + 1$, if the set of lengths of odd holes of G have k members, then G is $(k + 2)$-colorable.

Open problems

Question [Xu, 2024+]

Let $r \geq 2$ be an integer, and let $\mathcal{H}_r$ be the set of graphs with girth at least $2r$ which has no even hole of length at least $2r+2$. What is the smallest integer c_r such that $\chi(G) \leq c_r$ for every graph $G \in \mathcal{H}_r$? Is it true that $c_2 = 3$?

Thank you!